Journal of Engineering Mathematics 19 (1985) 45–55.

On extraneous solutions with uniformly bounded difference quotients for a discrete analogy of a nonlinear ordinary boundary-value problem *

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(Received October 22, 1984)

Summary

For a discrete analogy with $n-1 \in \mathbb{N}$ grid points of a nonlinear ordinary boundary-value problem with an implicit differential equation of the second order, the existence of $2^{n-1}-2$ extraneous solutions is shown, whose sequences of difference quotients of the first and the second order are uniformly bounded as $n \to \infty$. For selected explicitly represented sequences of extraneous solutions, the limiting function as $n \to \infty$ is explicitly given. These functions either do not solve the differential equation or only in a non-classical sense.

1. Introduction

Nonlinear ordinary bvp (= boundary-value problems) are considered on intervals $[a, b]$ of the independent variable x. These bvp consist of a differential equation of order m and m boundary conditions. If it exists, a classical solution of the bvp is denoted by u^*. Any consistent discretization of the bvp is adopted which makes use of the grid points $x_i \in [a, b]$ with $i = 1(1)n$, $n \in \mathbb{N}$, and $x_j \neq x_i$ for $j \neq i$. If it exists, an exact solution of this finite-dimensional approximation $F_n(u_n) = 0$ is denoted by u_n^*, where $u_n^* \in \mathbb{R}^n$. By use of the difference quotients up to order m employed in the discretization of the bvp, sequences of difference quotients are defined for a given sequence of difference solutions. As $n \to \infty$, the pointwise convergence to continuous limiting functions $f_\nu : [a, b] \to \mathbb{R}$ with $\nu = 0(1)m$ is a necessary requirement for a meaningful sequence $\{u_n^*\}$ and the pertinent sequences of difference quotients up to order m. The limiting functions define a classical solution of the bvp if $f_{\nu i}$ is the ν-th derivative at $x = x_i$ of a function $\hat{u}^* : [a, b] \to \mathbb{R}$ such that $\hat{u}^*$ satisfies the differential equation and the boundary conditions. Stability of a difference method implies that the sequence $\{(F_n')^{-1}\}$ is uniformly bounded (as $n \to \infty$) in a neighborhood of a discretization of u^*, where F_n' is the Frechet-derivative of F_n. This property guarantees pointwise convergence of $\{u_n^*\}$ toward u^* provided u^* is known to exist. A practical computation of the sequence $\{(F_n')^{-1}\}$ is generally not possible, not even for bounded n, since the location of the classical solutions u^* is to be determined and thus unknown.

* This paper is dedicated to Professor Dr. J. Weissinger on the occasion of his 70th birthday.

Sequences of difference solutions which do not converge to a classical solution are called "extraneous". Special cases of such sequences communicated in literature pertain to explicit or implicit differential equations which are not unusual in any sense; these sequences possess the following properties:

(a) As $n \to \infty$, either $\{u_n^*\}$ or any sequence of pertinent difference quotients (up to order m) does not converge to a continuous limiting function f_ν on $[a, b]$; see examples in [2], [3], [4] where one extraneous sequence is given explicitly in each case for explicit or implicit differential equations of orders 2 or 4.

(b) The sequence of difference solutions converges to a function $u \in C^m[a, b]$ satisfying the boundary conditions. The sequences of difference quotients up to order m are uniformly bounded for $n \to \infty$; however, u does not solve the differential equation, not even approximately. See examples in [4] where one extraneous sequence is given explicitly in each case. In the example presented subsequently in this paper, there are $2^n - 2$ extraneous solutions for every $n \in \mathbb{N}$, all of which possess property (b).

The preceding discussion reveals that difference methods for nonlinear bvp are not necessarily "well-behaved" in practically relevant cases such that the computation of a few terms of a sequence $\{u_n^*\}$ is sufficient to imply the existence of a classical solution with values close to the ones of the computed difference solutions. This, however, is the basis of the justification of the vast majority of pertinent engineering computations.

The existence of extraneous difference solutions can be excluded if the sufficient conditions of theorems [2] are satisfied which guarantee that the number of classical solutions of the bvp is equal to the number of difference solutions for every $n \in \mathbb{N}$. These conditions are satisfied only for special classes of nonlinear bvp.

The example presented subsequently in this paper pertains to the implicit differential equation $(y'')^2 - 12y' = 0$ for $x \in [0, 1]$ with the boundary conditions $y(0) = 0$ and $y(1) = 7$. Since the differential equation belongs to the class $f(y'') - g(x, y, y') = 0$ of such equations, property (b) may also be expected to hold for suitable discrete analogies of that subset of differential equations in this class which does not admit an explicit solution for y''. Consequently, it is irrelevant that the bvp $y'' \pm \sqrt{12y'} = 0$ for $x \in [0,1]$ with $y(0) = 0$, $y(1) = 7$, and $y'(x) \geqslant 0$ does not possess extraneous difference solutions under the employment of the same difference quotients as in the case of the implicit differential equation. In every investigated case, a numerically approximated extraneous difference solution for the bvp under consideration possesses a sufficiently large domain of attraction for the practical execution of the Newton iteration method.

Differential equations $f(x'') - g(x, y, y') = 0$ appear in mathematical models of the following types: (A) in mechanics if mass depends on acceleration as, e.g., in the case of an oscillating solid body moving in an external compressible "heavy" fluid which also flows into or out of a duct inside the body or (B) in the case of LC-oscillators in electrical engineering if the inductivity depends on the derivative di/dt of the current i, due to a special control system.

Since ordinary bvp are special cases of elliptic bvp, the existence of extraneous difference solutions may also be expected in the case of suitable discrete analogies of certain nonlinear elliptic bvp. This existence is well known in the case of suitable implicit discretizations of nonlinear parabolic or hyperbolic initial/boundary-value problems, [1].

2. The boundary-value problem and approximations of its solutions via a discrete analogy

Real-valued classical solutions of the bvp

$$(y'')^2 - 12y' = 0, \quad y(0) = 0, \quad y(1) = 7 \tag{2.1}$$

will be considered. This example does not satisfy the conditions of theorems in [2]. The following discussions on extraneous solutions are (i) generalizations of the ones in [4] on an almost identical bvp and (ii) essentially presuppose a discretization of the implicit form of the differential equation as given in (2.1). The substitution $v := y'$ yields $v' = \pm\sqrt{12v}$ and, thus,

$$v(x) = \begin{cases} 3(x + c_0)^2 & \text{with } c_0 \in \mathbb{R}, \text{ or} \\ 0 \end{cases} \tag{2.2}$$

Therefore, the differential equation in (2.1) is solved *(a)* by the functions

$$y(x) = \left\{ \begin{matrix} (x + c_0)^3 + c_1 & \text{or} \\ c_2 & \end{matrix} \right\} \quad \text{where } c_1, c_2 \in R, \tag{2.3}$$

and (b) arbitrary twice continuously differentiable compositions of these functions. Only the following functions are classical solutions of the differential equation *and* the boundary conditions:

$$\begin{aligned} &y_{\mathrm{I}}(x) := (x + 1)^3 - 1, \\ &y_{\mathrm{II}}(x) := (x - 2)^3 + 8 \Rightarrow y_{\mathrm{I}} < y_{\mathrm{II}}, \\ &0 < x < 1. \end{aligned} \tag{2.4}$$

The following discrete analogy of (2.1) will be investigated which possesses second order of consistency:

$$\begin{aligned} &\left((y_{j+1} - 2y_j + y_{j-1})/h^2\right)^2 = 12(y_{j+1} - y_{j-1})/2h \\ &\quad \text{for } j = 1(1)n - 1, \\ &y_0 = 0, \quad y_n = 7, \quad hn = 1, \quad n \in \mathbb{N}, \quad n \geqslant 2. \end{aligned} \tag{2.5}$$

It will be shown that (2.5) possesses precisely 2^{n-1} real-valued solutions. For this purpose, the problem is equivalently represented via

$$u_j := y_j - y_{j-1} \quad \text{for } j = 1(1)n \Rightarrow y_m = y_m - y_0 = \sum_{i=1}^{m} u_i, \quad m = 1(1)n, \tag{2.6}$$

thus yielding

$$\left((u_{j+1} - u_j)/h^2\right)^2 = 6(u_{j+1} + u_i)/h \qquad \text{for } j = 1(1)n - 1 \tag{2.7a}$$

and

$$\sum_{j=1}^{n} u_j = 7, \quad u_1 = y_1 - y_0 = y_1 \text{ is free.} \tag{2.7b}$$

According to (2.7a),

$$u_{j+1} = u_j + k \pm \sqrt{k^2 + 4ku_j} \qquad \text{for} \quad j = 1(1)n-1, \quad \text{where } k := 3h^3. \tag{2.8}$$

Provided u_1 has been selected, independent choice of the plus sign or the minus sign are admissible in the subsequent computation of $u_2, u_3, \ldots$. Each fixed choice for $j = 2(1)n$ will be denoted as a "sign-pattern" of (2.8). For an arbitrary pattern and any $n \in \mathbb{N}$ fixed, it is not clear a priori whether or not (2.7b), and thus $y_n = 7$, can be satisfied via any choice or even a unique choice of $u_1 \in \mathbb{R}$.

At first, the case of the "plus sign only" in (2.8) is considered, i.e., the choice of $+\sqrt{\cdot}$ for $j = 1(1)n-1$. Due to $k > 0$, (i) u_{j+1} is a monotonically increasing function of u_j and (ii) so is $y_n = \sum_{j=1}^{n} u_j$, i.e., there is at most one solution satisfying (2.7b). In order to show the existence of a number $u_1 \in \mathbb{R}$ such that (2.7b) is satisfied, u_1 is expressed via a free parameter $q \in \mathbb{R}$:

$$u_1 = q(q-1)k, \tag{2.9}$$

where values $u_1 \geqslant -k/4$, ensuring real solutions, already are obtained under the restriction $q \geqslant 1/2$. By induction, (2.8) and (2.9) yield

$$u_{j+1}(n, q) = (q + j - 1)(q + j)k \qquad \text{for} \quad j = 1(1)n-1 \tag{2.10}$$

and, due to (2.7b), the condition, denoted by $\overset{!}{=}$,

$$y_n = y_n(n, q(n)) = \sum_{j=1}^{n} u_j(n, q(n)) = n\left(q^2 + q(n-2) + (n-2)(n-1)/3\right)3/n^3 \overset{!}{=} 7. \tag{2.11}$$

This quadratic equation for $q = q(n)$ possesses the root

$$q = \left(2 - n + \sqrt{9n^2 + 4/3}\right)/2 \tag{2.12}$$

where the case of the minus-sign of the root is not of interest since $q \geqslant 1/2$. The solution (2.10) satisfying (2.11) will be denoted by $u^{(+)}$ or $u_j^{(+)}$ for $j = 1(1)n$. For rational $x = m/n \in (0, 1]$, this solution yields

$$y_j(n, q(n)) = \sum_{j=1}^{m=nx} u_j^{(+)}(n, q(n)) = \frac{3}{n^3} xn\left(q^2 + q(xn-2) + (xn-2)(xn-1)/3\right). \tag{2.13}$$

Correspondingly, there follows the uniform pointwise convergence

$$\lim_{n\to\infty} y_j(n, q(n)) = y_{\mathrm{I}}(x) \quad \text{for every } x = m/n \in [0, 1]. \tag{2.14}$$

Next, the choice of the "minus sign only" in (2.8) is considered i.e., $-\sqrt{\cdot}$ for $j = 1(1)n-1$. This yields a solution $u_j^{(-)}(n, q(n))$ denoted by $u^{(-)}$ which satisfies

$$u_j^{(-)} = u_{n+1-j}^{(+)} \qquad \text{for} \quad j = 1(1)n. \tag{2.15}$$

For rational $x \in [0, 1]$, the corresponding solution $y_j^{(-)}(n, q(n))$ converges in a pointwise sense to $y_{\mathrm{II}}(x)$. In numerical experiments, y_{I} and y_{II} have been approximated iteratively by use of the Newton method as applied to (2.5), starting within a finite domain of attraction centered with respect to these functions such that the components of these vectors were chosen by a random number generator.

3. Extraneous solutions

It will be shown now that (i) there exists precisely one solution $u_j(n, q(n))$ for every sign pattern in (2.8), (ii) for each fixed n, each such solution is bounded according to (3.5), (iii) these solutions are distinct, and (iv) the difference quotients of the first and the second order of these solutions are uniformly bounded as $n \to \infty$. The verification of (i) – (iv) proceeds via nine steps, (I)–(IX), to be presented now.

(I) *For any fixed* $j \in \{1, \ldots, n-1\}$, (2.8) *yields* $du_{j+1}/du_j > 0$ (*i*) *in the case of the plus sign and* (*ii*) *in the case of the minus sign provided that then* $u_j > 3k/4$.

(II) *For* $u_1 \geqslant -k/4$ *and every sign pattern*, u_j *is real-valued for* $j = 1(1)n$, which obviously is correct in the case of the "plus sign only" in (2.8). The representation (2.9) is now used for u_j,

$$u_j = (q-1)qk \quad \text{with } q \geqslant 1/2 \tag{3.1}$$

since this restriction of q already yields every $u_j \in \mathbb{R}$ such that $u_{j+1} \in \mathbb{R}$. Due to (2.8),

$$u_{j+1} = \left((q-1)q + 1 \pm \sqrt{1 + 4(q-1)q}\right)k = \left(q^2 - q + 1 \pm |2q-1|\right)k, \tag{3.2}$$

where $|2q-1| = 2q-1$ due to $q \geqslant 1/2$. If the sign pattern prescribes the plus sign for the subscript $j+1$ under consideration, then

$$u_{j+1} = q(q+1)k > -k/4 \tag{3.3a}$$

because of $q \geqslant 1/2$. If the minus sign is prescribed for this j, then

$$u_{j+1} = (q-2)(q-1)k \geqslant -k/4 \tag{3.3b}$$

since $(q-2)(q-1)$ possesses the minimum value $-k/4$ for $q = 3/2$. The representation

(3.1) for u_{j+1} is obtained via the substitutions $\hat{q} := q + 1$ in the case of (3.3a) and $q \geqslant 3/2$ in the case of (3.3b) or $\hat{q} := 2 - q \geqslant 1/2$ for $1/2 \leqslant q < 3/2$ in the case of (3.3b).

(III) *With* $u_j > -k/4$ *fixed, the plus sign in* (2.8) *yields a larger value of* u_{j+1} *than the minus sign.*

(IV) *For arbitrary sign patterns,* u_{j+1} *is a monotonically increasing function of* u_j *for* $j = 1(1)n - 1$. According to (I), this montonicity holds if $u_j > 3k/4$. It is assumed that there exists a special sign pattern and a pertinent solution of (2.7) such that there is a $j_0 \in \{1,\ldots,n\}$ with $u_{j_0} \leqslant 2k$. There holds $u_{j_0} < u_1^{(+)}$ for every choice of $n \geqslant 2$ as is easily verified via an inspection of (2.9) and (2.12). This implies (i) $u_{j_0+p} \geqslant u_{1+p}^{(+)}$ for $p = 1(1)n - j_0$ since the "plus sign only" in (2.8) yields the maximum rate of increase. Because of (I), $u_{j_0} < u_1^{(+)} < u_{1+n-j_0}^{(+)}$ and, analogously, (ii) $u_{j_0-p} < u_{1+n-j_0+p}^{(+)}$ for $p = 1(1)j_0 - 1$. Due to (i) and (ii),

$$\sum_{j=1}^{n} u_j < \sum_{j=1}^{n} u_{j_0}^{(+)} = 7. \tag{3.4}$$

(V) *For every sign pattern and* $n \in \mathbb{N}$, *there is precisely one solution of* (2.7*a*, *b*) with (IV) implying uniqueness of each solution. Here the existence of the solutions follows from (i) $y_n < 7$ if $u_1 \leqslant 2k$ due to (IV), (ii) $y_n \to \infty$ as $u_1 \to \infty$ due to (IV), and (iii) the continuous dependency on u_1 of $y_n = \Sigma_{j=1}^{n} u_j$.

(VI) *For any fixed* $n \in \mathbb{N}$, *different sign patterns imply different solutions.* Two sign patterns are said to be different if they do not coincide at least for one subscript j_0 with $2 \leqslant j_0 \leqslant n$. Two solutions u and v are said to be different if there exists a subscript j such that $u_j \neq v_j$. If j_0 is a subscript such that two sign patterns are different and if there holds $u_{j_0-1} = v_{j_0-1} > 2k$ for the pertinent solutions due to (IV), then $u_{j_0} \neq v_{j_0}$, according to (III). As a corollary, it is observed that there are precisely 2^{n-1} different solutions since there are $n - 1$ grid points with choices of the sign of the root in (2.12). As n increases, so does the number 2^{n-1} of sequences of solutions of (2.5).

(VII) *Every sequence* $\{u_j\}$ *solving* (2.7*a*, *b*) *is enclosed as follows*:

$$\sum_{j=1}^{m} u_j^{(+)} \leqslant \sum_{j=1}^{m} u_j \leqslant \sum_{j=1}^{m} u_j^{(-)} \qquad \text{for } m = 1(1)n, \text{ under the admission of every possible sign pattern.} \tag{3.5}$$

If the first one of these two inequalities is not true, then there exists a smallest number $\hat{m} \in \{1,\ldots,n\}$ such that

$$\left.\begin{aligned} &\sum_{j=1}^{\hat{m}-1} u_j^{(+)} \leqslant \sum_{j=1}^{\hat{m}-1} u_j \qquad \text{if } \hat{m} > 1 \qquad (3.6\text{a})\\ &\text{and}\\ &\sum_{j=1}^{\hat{m}} u_j^{(+)} > \sum_{j=1}^{\hat{m}} u_j \qquad (3.6\text{b}) \end{aligned}\right\} \Rightarrow u_{\hat{m}}^{(+)} > u_{\hat{m}}.$$

This implies that

$$u_j < u_j^{(+)} \qquad \text{for} \quad j = \hat{m} + 1(1)n \tag{3.7}$$

for every sign pattern since the choice of the "plus sign only" in (2.8) ensures the maximum rate of increase of the sequence $\{u_j\}$ for $j > \hat{m}$. Because of (3.6b) and (3.7), there follows $\Sigma_{j=1}^{n} u_j < 7$. The second inequality in (3.5) is verified analogously.

(VIII) *For $n - 1 \in \mathbb{N}$, the difference quotients $(y_j - y_{j-1})/h$ are uniformly bounded as $n \to \infty$.* Because of (2.6), bounds of u_j will be determined first. A lower bound $u_j > 2k$ for every $j = 1(1)n$ follows from (IV). Due to (2.7b), u_{j_0} with any $j_0 \in \{1,\dots,n\}$ is maximized if the $n-1$ terms other than u_j are minimized in that sum. According to the expressions for u_{j_0+1} and u_{j_0-1} following from (3.3a, b) such that $j = j_0$ is fixed, u_j with $j \neq j_0$ is minimized if (i) the "minus sign only" is used for $j \geqslant j_0$ and (ii) the "plus sign only" is used for $j < j_0$. Consequently, the maximum possible value of u_{j_0} is attained for $j_0 = 0$ or $j_0 = n$. Without a loss of generality, the case of $j_0 = n$ will be discussed further. Due to (III), the choice of the "plus sign only" in (2.8) yields the maximum increase of the sequence $u_1, u_2, \dots, u_n$ and thus the desired minimization of the sum $u_1 + u_2 + \dots + u_{n-1}$. Correspondingly, $u_n^{(+)}$ is the maximum element, which satisfies

$$u_n^{(+)} = (q + n - 2)(q + n - 1)k, \tag{3.8}$$

making use of (2.10). Therefore and because of (2.12) and $k = 3n^{-3}$,

$$2k/h < u_j/h \leqslant u_n^{(+)}/h = nu_n^{(+)}$$

$$= n\left(n - 2 + \frac{2 - n + \sqrt{9n^2 + 4/3}}{2}\right)\left(n - 1 + \frac{2 - n + \sqrt{9n^2 + 4/3}}{2}\right)\frac{3}{n^3}. \tag{3.9}$$

Therefore, a lower bound and a crude upper bound, M, can be defined as follows:

$$0 < \frac{y_j - y_{j-1}}{h} = \frac{u_j}{h} < M = \frac{3}{n^2}\left(n + \frac{4n}{2}\right)^2 = 27 \qquad \text{for } n \geqslant 2 \text{ and } j = 1(1)n. \tag{3.10}$$

(IX) *For $n - 1 \in \mathbb{N}$, the difference quotient $(y_{j+1} - 2y_j + y_{j-1})/h^2$ is uniformly bounded as $n \to \infty$ since*

$$\left|\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2}\right| = \left|\frac{u_{j+1} - u_j}{h^2}\right| = \sqrt{12\frac{u_{j+1} + u_j}{2h}} =: A_j \qquad \text{for } j = 1(1)n - 1, \tag{3.11}$$

due to (2.6) and (2.7a). Because of (3.10), the following crude bound is obtained:

$$0 < A_j < \sqrt{6(27 + 27)} = 18 \qquad \text{for } j = 1(1)n \text{ and } n \geqslant 2. \tag{3.12}$$

(VIIIa) *The central first difference quotient is bounded as follows:*

$$\left|\frac{y_{j+1} - y_{j-1}}{2h}\right| \leqslant \frac{A_j^2}{12} < \frac{18}{12} = 27, \tag{3.13}$$

due to (2.5), (3.11), and (3.12).

4. On limiting functions of sequences of extraneous solutions

For any fixed $\hat{n} \in \mathbb{N}$, a solution of (2.7a, b) is obtained via the choices of (i) an arbitrary sign pattern in (2.8) and (ii) a suitably determined value of u_1 such that (2.7b) is satisfied, which is always possible, according to (V). For $n \to \infty$, pointwise convergence has already been shown in the cases of the sequences (a) $\{y^{(+)}\}$ to the limiting function y_{I} and (b) $\{y^{(-)}\}$ to the limiting function y_{II}. Subsequently, convergence and pertinent limiting functions will be discussed for one sign pattern in the case (A) and sign patterns admitting a wide scope of variations in the case of (B).

(A) Here, the alternating sign pattern $+, -, +, -, \dots$ is chosen in (2.8). Then, (2.8) is satisfied by

$$u_{2j} = (q-1)qk \qquad \text{for } j = 1(1)m,$$

$$u_{2j+1} = (q-2)(q-1)k \qquad \text{for } j = 0(1)m-1,$$

$$2m = n \qquad \text{if } n \text{ is even.} \tag{4.1}$$

The condition (2.7b) requires that there holds

$$\sum_{j=1}^{m} u_{2j} + \sum_{j=0}^{m-1} u_{2j+1} = 7; \tag{4.2}$$

this yields the quadratic equation $q^2 - 2q + 1 - 7/2km = 0$ with $2m = n$, whose positive root is $q = 1 + \sqrt{56m^2/6}$. Therefore, and due to (4.1), there follows

$$u_{2j} = 7h + \sqrt{21}\,h^2 \quad \text{and} \quad u_{2j+1} = 7h - \sqrt{21}\,h^2. \tag{4.3}$$

Analogous to (2.13), $x = \hat{m}/m \in (0, 1)$ with $\hat{m} = 1(1)m$ yields

$$\lim_{m \to \infty} \left(\sum_{j=0}^{\hat{m}=mx} u_{2j+1} + \sum_{j=1}^{\hat{m}=mx} u_{2j} \right) = 7x =: \varphi(x), \tag{4.4}$$

which is not a solution of the differential equation $(y'')^2 - 12y' = 0$, i.e., (4.1) generates a sequence of extraneous solutions of (2.7a, b).

Due to (2.6), there holds

$$y_{2j} = 7 \cdot 2jh \text{ for } j = 1(1)m \text{ and } y_{2j+1} = 7 \cdot (2j+1)h - \sqrt{21}\,h^2 \text{ for } j = 0(1)m-1, \tag{4.5}$$

which is an approximation of $\varphi(x)$. Consequently,

$$\frac{y_{2j+1} - y_{2j-1}}{2h} = \frac{y_{2j} - y_{2j-2}}{2h} = 7 = \varphi'(x),$$

$$\frac{y_{2j} - 2_{j-1}}{h} = 7 + h\sqrt{21}, \tag{4.6}$$

which implies that $M = 7 + \epsilon$, with an arbitrarily small $\epsilon \in \mathbb{R}^+$, is the value of the bound in (3.10) holding here. Additionally,

$$\frac{y_{2j+2} - y_{2j+1} + y_{2j}}{h^2} = 2\sqrt{21}\,, \quad \frac{y_{2j+1} - y_{2j} + y_{2j-1}}{h^2} = -2\sqrt{21}$$

$$\text{for} \quad j = 1(1)\frac{n}{2} - 1 \text{ if } n \text{ is even.} \tag{4.7}$$

which implies that there holds $A_j = 2\sqrt{21} \approx 9.1$ in this special case. Due to numerical experiments, making use of the Newton iteration method, the solution (4.5) possesses a finite domain of attraction.

REMARKS: (1) For n odd, the solution is given by the expressions for u_{2j} and u_{2j+1} in (4.3) provided these expressions are multiplied by $7 \cdot (7 - h^2\sqrt{21})^{-1}$. (2) It can be shown that the following sign patterns in (2.8) yield the limiting function $7x$ provided the n-dependent number u_1 is chosen suitably: (1) +, +, −, −, +, +,..., (2) +, +, +, −, −, −, +, +, +,..., etc.

(B1) Here, the following sign pattern is chosen:

$$\text{"+ only"} \quad \text{for} \quad j = 1(1)m \text{ with } m := n/2 \quad \text{and}$$

$$\text{"− only"} \quad \text{for} \quad j = m + 1(1)n \text{ if } n \text{ is even.} \tag{4.8}$$

Then, (2.8) is satisfied by

$$u_j = (q + j - 2)(q + j - 1)k \qquad \text{for} \quad j = 1(1)m \quad \text{and}$$

$$u_{m+s} = (q + m - 2 - s)(q + m - 1 - s)k \qquad \text{for} \quad s = 1(1)m. \tag{4.9}$$

Analogous to (4.2), there follows a quadratic equation for q whose positive root is

$$q = \tfrac{1}{2}\left(3 - m + \sqrt{37m^2 + 1/3}\right). \tag{4.10}$$

Analogous to (2.13), $x := \hat{m}/2m \in (0, 1/2]$ yields a sequence of approximations for $y_j(n, q(n))$ which converges in a pointwise sense to the limiting function

$$y^{(L)}(x) := \left(x + \frac{-1 + \sqrt{37}}{4}\right)^3 - \left(\frac{-1 + \sqrt{37}}{4}\right)^3 \qquad \text{for} \quad x \in [0, 1/2]. \tag{4.11}$$

Analogously, $x := \hat{m}/2m \in (1/2, 1]$ yields the limiting function

$$y^{(R)}(x) := \left(x + \frac{-3 - \sqrt{37}}{4}\right)^3 + 7 - \left(\frac{1 - \sqrt{37}}{4}\right)^3 \qquad \text{for} \quad x \in (1/2, 1]. \tag{4.12}$$

These functions satisfy

$$y^{(L)\prime\prime}(1/2) = -y^{(R)\prime\prime}(1/2) = \tfrac{6}{4}(1 + \sqrt{37}) \tag{4.13}$$

and the following relaxation of the bvp (2.1):

$$\left(y^{(L)\prime\prime}\right)^2 - 12\,y^{(L)\prime} = 0 \qquad \text{for} \quad x \in (0, 1/2),$$

$$\left(y^{(R)\prime\prime}\right)^2 - 12\,y^{(R)\prime} = 0 \qquad \text{for} \quad x \in (1/2, 1),$$

$$y^{(L)}(0) = 0, \quad y^{(R)}(1) = 7, \quad y^{(L)}(1/2) = y^{(R)}(1/2),$$

$$y^{(L)\prime}(1/2) = y^{(R)\prime}(1/2), \tag{4.14}$$

making use of one-sided derivatives of the second order at $x = 1/2$. Therefore, (i) the composition of $y^{(L)}$ and $y^{(R)}$ represents a nonclassical generalized solution of the bvp (2.1) and (ii) the sequence of solutions defined by (4.8) and (4.9) is extraneous.

REMARK: The choices "-only" for $j = 1(1)m$ and "+only" for $j = m + 1(1)n$ yield limiting functions $\hat{y}^{(L)}$ and $\hat{y}^{(R)}$ which deviate from $y^{(L)}$ and $y^{(R)}$ only via a replacement of the sign of the roots in (4.11), (4.12) by the opposite sign.

(B2) Here, the following sign pattern is chosen under the assumption of $n/4 \in \mathbb{N}$:

$$\text{"+only"} \qquad \text{for} \quad j = 1(1)m \text{ with } m := n/2 \quad \text{and}$$

$$\text{the alternating sign pattern } +, -, +, -, \ldots \qquad \text{for} \quad j = m + 1(1)n. \tag{4.15}$$

Analogous to (2.10) or (4.1), respectively, (2.8) is satisfied by

$$u_j = (q + j - 2)(q + j - 1)k \qquad \text{for} \quad j = 1(1)m \tag{4.16}$$

and

$$u_{m+2j} = (q + m - 2)(q + m - 1)k \quad \text{and}$$

$$u_{m+2j-1} = (q + m - 1)(q + m)k \qquad \text{for} \quad j = 1(1)\frac{m}{2}. \tag{4.17}$$

The condition (2.7b) yields a quadratic equation for q whose positive solution is

$$q = \tfrac{1}{4}\left(4 - 3m + \sqrt{9m^2 - 24m + 16 - \tfrac{8}{3}(-52m^2 - 9m + 15)}\,\right). \tag{4.18}$$

Analogous to (2.13), $x = \hat{m}/2m \in (0, 1/2]$ yields an expression for $y_j(n, q(n))$ which converges in a pointwise sense to the limiting function

$$y^{(L)}(x) := x^3 + \frac{-9 + \sqrt{1329}}{8}x^2 + \frac{470 - 6\sqrt{1329}}{64}x \qquad \text{for} \quad x \in [0, 1/2]. \tag{4.19}$$

Correspondingly, $x = \frac{1}{2} + \hat{m}/2m \in (1/2, 1]$ yields the limiting function

$$y^{(R)}(x) := \frac{1}{32}(1 - \sqrt{1329}\,) + \frac{223 + \sqrt{1329}}{32}x \qquad \text{for} \quad x \in [1/2, 1]. \tag{4.20}$$

Whereas $y^{(L)(i)}(1/2) = y^{(R)(i)}(1/2)$ for $i = 0$ or 1, $y^{(L)''}(1/2) \neq y^{(R)''}(1/2) = 0$. Clearly, (i) the composition of $y^{(L)}$ and $y^{(R)}$ does not yield a solution of the differential equation $(y'')^2 - 12y' = 0$ and (ii) the sequence of solutions defined by (4.15) and (4.16) is extraneous.

REMARK: The analysis of the case (B2) can be repeated for modifications of (4.15) such that $n/2$ is replaced by, e.g., $n/4$.

For each point in the set

$$\{(x, y) \mid 0 \leqslant x \leqslant 1,\ y_{\mathrm{I}}(x) \leqslant y \leqslant y_{\mathrm{II}}(x)\} \subset \mathbb{R}^2, \tag{4.21}$$

it can be shown that there is at least one limiting function of a sequence of extraneous solutions of (2.7a, b) taking on the value y at this point x.

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